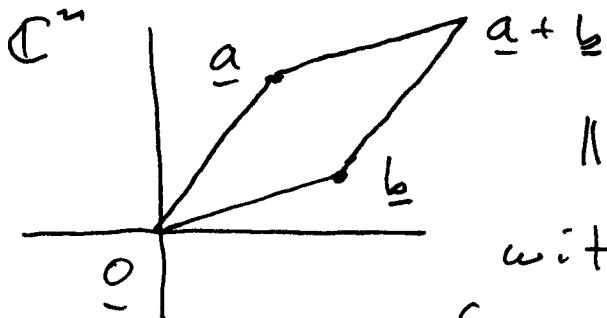


FTA (fundamental theorem of algebra).

A nonconstant polynomial p , with real or complex coefficients, has a complex zero.

Triangle Inequality.



$$\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|,$$

with equality iff one of \underline{a} or \underline{b} is a nonnegative scalar multiple of the other.

□ Same as $\|\underline{a} + \underline{b}\|^2 \leq (\|\underline{a}\| + \|\underline{b}\|)^2$, that is $(\underline{a} + \underline{b})^H(\underline{a} + \underline{b}) = (\underline{a}^H + \underline{b}^H)(\underline{a} + \underline{b}) \leq \|\underline{a}\|^2 + 2\|\underline{a}\|(\|\underline{b}\| + \|\underline{b}\|^2)$, that is
 ~~$\underline{a}^H \underline{a} + \underline{a}^H \underline{b} + \underline{b}^H \underline{a} + \underline{b}^H \underline{b} \leq \|\underline{a}\|^2 + 2\|\underline{a}\|(\|\underline{b}\| + \|\underline{b}\|^2)$~~ , that is, since $\underline{b}^H \underline{a} = \overline{\underline{a}^H \underline{b}}$,
 ~~$2\operatorname{Re} \underline{a}^H \underline{b} \leq 2\|\underline{a}\| \|\underline{b}\|$~~ . But $\operatorname{Re} z \leq |z|$ for any complex number z , so $\operatorname{Re} \underline{a}^H \underline{b} = |\underline{a}^H \underline{b}| \leq \|\underline{a}\| \|\underline{b}\|$ by Cauchy's inequality. For equality we must have $\underline{a}^H \underline{b} \geq 0$. ■

$$\underline{b} \leftarrow \underline{b} - \underline{a} \Rightarrow \|\underline{b}\| - \|\underline{a}\| \leq \|\underline{b} - \underline{a}\|.$$

$$\text{Swap } \underline{a} + \underline{b} \Rightarrow \|\underline{a}\| - \|\underline{b}\| \leq \|\underline{b} - \underline{a}\|.$$

$$\text{All together: } |\|\underline{b}\| - \|\underline{a}\|| \leq \|\underline{b} - \underline{a}\|.$$

$$\underline{a} \leftrightarrow -\underline{a} \Rightarrow |\|\underline{a}\| - \|\underline{b}\|| \leq \|\underline{a} + \underline{b}\|.$$

All together:

$$|\|\underline{a}\| - \|\underline{b}\|| \leq \|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|.$$

Also with three vectors,

$$|\|\underline{a}\| - \|\underline{b}\| - \|\underline{c}\|| \leq \|\underline{a} + \underline{b} + \underline{c}\| \leq \|\underline{a}\| + \|\underline{b}\| + \|\underline{c}\|,$$

etcetera.

It goes without saying that these triangle inequalities also hold for $C = C^1$!

The FTA. We consider monic polynomials

$$P(z) = a_0 + a_1 z + \dots + a_n z^n, \quad a_n := 1$$

It's easy to scale z so the coefficients satisfy $|a_k| \leq 1$, $k = 0, 1, \dots, n$. For $\alpha > 0$ let

$$g(z) := \alpha^n P\left(\frac{z}{\alpha}\right) =$$

$$= \alpha^n a_0 + \alpha^{n-1} a_1 z + \cdots + \\ + \alpha a_{n-1} z^{n-1} + a_n z^n.$$

If z_0 is a zero of q then z_0/α is a zero of p , and vice versa.

Just take

$$\alpha \leq \min \left\{ \frac{1}{|a_0|^{\frac{1}{n}}}, \frac{1}{|a_1|^{\frac{1}{n-1}}}, \dots, \frac{1}{|a_{n-1}|} \right\}$$

so the coefficients b_k of q have all $|b_k| \leq 1$. Note $b_n = a_n = 1$!

Theorem 1 (a little theorem). If $n \geq 1$, $a_n = 1$, $|a_k| \leq 1$ for $k = 0, 1, \dots, n-1$, then $|p(z)| \geq 1 > 0$ for $|z| \geq 2$. That is, the zeros of p , if any, lie in the open disk $|z| < 2$.

□
$$p(z) = z^n \left(1 - \frac{a_{n-1}}{z} - \cdots - \frac{a_0}{z^n} \right).$$

By the triangle inequality, and $|a_k| \leq 1$, and $\frac{1}{|z|} \leq \frac{1}{2}$,

$$|p(z)| \geq |z|^n \left(1 - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_0|}{|z|^n} \right) \\ \geq |z|^n \left(1 - \frac{1}{2} - \cdots - \frac{1}{2^n} \right)$$

$$\begin{aligned}
 &= |z|^n \left(1 - \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} \right) \right)^{\text{FTA 4}} \\
 &= |z|^n \left(1 - \cancel{\frac{1}{2}} \frac{1 - \frac{1}{2^n}}{1 - \cancel{\frac{1}{2}}} \right) \\
 &= \left(\frac{|z|}{2} \right)^n \geq 1 > 0. \quad \blacksquare
 \end{aligned}$$

We can also shift the origin so that $a_{n-1} = 0$. By the binomial theorem

$$\begin{aligned}
 P(z+s) &= (z+s)^n + \\
 &\quad + a_{n-1}(z+s)^{n-1} + \text{"lot"} \\
 &= z^n + ns z^{n-1} + \text{lot} \\
 &\quad + a_{n-1} z^{n-1} + \text{lot} \\
 &= z^n + (ns + a_{n-1}) z^{n-1} + \text{lot} \\
 &= z^n + b_{n-2} z^{n-2} + \dots + b_0
 \end{aligned}$$

provided we choose the shift s to be

$$s = -\frac{a_{n-1}}{n}.$$

This Cov will make the sum of the zeros = 0.

Theorem 2 (only a little bigger theorem). If $n \geq 1$, $a_n = 1$, $|a_k| \leq 1$ for $k = 0, 1, \dots, n$, and $a_{n-1} = 0$, then $|P(z)| \geq g > 0$ for $|z| \geq g$, with $g = \frac{1+\sqrt{5}}{2}$ the golden mean.

□ g satisfies $g^2 - g - 1 = 0$, so

$$\frac{1}{g-1} = g. \quad \text{In fact } x^2 - x - 1 = (x-g)(x+\frac{1}{g})$$
 ~~$\Rightarrow x^2 - x + 1 > 0$ for $x > g$.~~ Now

$$|z| \geq g \Rightarrow$$

$$\begin{aligned}
 |P(z)| &\geq |z|^n \left(1 - \frac{1}{g^2} - \dots - \frac{1}{g^n} \right) \\
 &= |z|^n \left(1 - \frac{1}{g^2} \left(1 + \frac{1}{g} + \dots + \frac{1}{g^{n-2}} \right) \right) \\
 &= |z|^n \left(1 - \frac{1}{g^2} \frac{1 - \frac{1}{g^{n-1}}}{1 - \frac{1}{g}} \right) \\
 &= |z|^n \left(1 - \frac{\frac{1}{g} - \frac{1}{g^n}}{g-1} \right) \\
 &= |z|^n \underbrace{\frac{g-1 - \frac{1}{g^{n-1}}}{g-1}}_{\frac{g-1}{g} + \frac{1}{g^n}} \\
 &= g \left(\frac{|z|}{g} \right)^n \geq g. \blacksquare
 \end{aligned}$$

Proof of the FTA. Suppose all $|a_k| \leq 1$, so there are no zeros in $|z| \geq 2$ (or in $|z| \geq g$ if $a_{n+1} = 0$). On the closed disk $|z| \leq 2$ (or $|z| \leq g$) the function $|p(z)|$ is continuous. It thus assumes its minimum value: $|p(z_0)| = \min \{|p(z)| : |z| \leq 2\}$. If $w_0 := p(z_0) = 0$ we are done: z_0 is a (complex) zero of p .

Now assume $|w_0| > 0$ to obtain a contradiction. Also let $w = p(z)$. Let $\varepsilon > 0$. We're going to choose ε "sufficiently small". Let $z = z_0 + \varepsilon e^{i\theta}$. As θ runs from 0 to 2π , z runs over the (tiny) circle with center z_0 and radius ε , exactly once. We study the curve $w = p(z) = p(z_0 + \varepsilon e^{i\theta})$, $0 \leq \theta \leq 2\pi$. By

Taylor expansion of p about z_0 ,

$$w = p(z)$$

$$= w_0 + c_1(z - z_0) + \dots + c_n(z - z_0)^n$$

with

$$c_k = \frac{p^{(k)}(z_0)}{k!}, \quad k = 0, 1, \dots, n.$$

Since p is nonconstant there is a first c_k , call it $c := c_x$, with $c_k \neq 0$. Then, with $z_1 = z_0 + \varepsilon e^{i\theta}$,

$$w = w_0 + c \varepsilon^x e^{ix\theta} + \dots + c_n \varepsilon^n e^{inx\theta}$$

$$= w_0 + c \varepsilon^x e^{ix\theta}.$$

$$\underbrace{\cdot \left\{ 1 + \frac{c_{x+1}}{c} \varepsilon e^{i\theta} + \dots + \frac{c_n}{c} \varepsilon^{n-x} e^{i(n-x)\theta} \right\}}_{=: O(\varepsilon)}.$$

Let

$$\gamma := \max \left\{ \left| \frac{c_k}{c} \right| : k < k \leq n \right\}.$$

Then, by the triangle inequality,

$$|O(\varepsilon)| \leq \gamma (\varepsilon + \varepsilon^2 + \dots + \varepsilon^{n-x})$$

$$\leq \gamma (\varepsilon + \varepsilon^2 + \dots)$$

$$= \frac{\gamma \varepsilon}{1-\varepsilon} \text{ provided } \varepsilon < 1.$$

One of the conditions we want on ε is that $|O(\varepsilon)| < 1$. This will be true if $\frac{\gamma \varepsilon}{1-\varepsilon} < 1$, that is if $\gamma \varepsilon < 1 - \varepsilon$, or $(1+\gamma)\varepsilon < 1$, that is

$$\boxed{\varepsilon < \frac{1}{1+\gamma}}, \text{ Then also } \varepsilon < 1. \text{ This}$$

guarantees that $|w-w_0| = |c|\varepsilon^k|1+O(\varepsilon)| \geq |c|\varepsilon^k(1-|O(\varepsilon)|) > 0$ on the "w-curve".

Finally, we also want $|w-w_0| < |w_0|$ on the w-curve. Since $\varepsilon < \frac{1}{1+\gamma} \Rightarrow$

$|O(\varepsilon)| < 1$ then $|w-w_0| < 2|c|\varepsilon^k$, so

$$|w-w_0| < |w_0| \text{ if, also, } \varepsilon \leq \left(\frac{|w_0|}{2|c|}\right)^{\frac{1}{k}}$$

Bottom line: $\varepsilon < \min\left\{\frac{1}{1+\gamma}, \left(\frac{|w_0|}{2|c|}\right)^{\frac{1}{k}}\right\}$ guarantees that

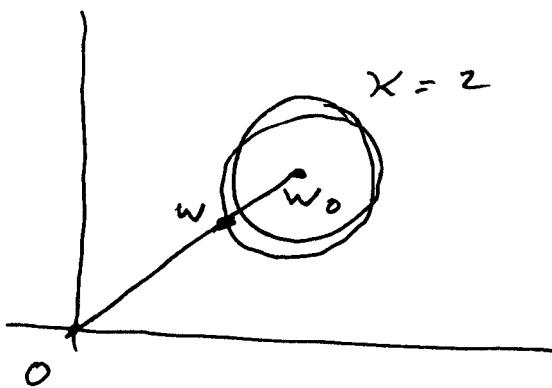
$$0 < |w-w_0| < |w_0|$$

on the w-curve. For all such small ε , as θ runs from 0 to 2π , $w = p(z_0 + \varepsilon e^{i\theta})$ winds around the point $w_0 = p(z_0)$ $k \geq 1$ times.

w is never $= w_0$ or $= 0$, and the distance between w and w_0

is smaller than the distance between w_0 and 0. It follows

(1)



that there's a point w on the w -curve with $0 < |w| < |w_0|$.

This contradiction to the choice of w_0 completes one proof of the FTA ■

Remarks. This nonconstructive proof of Gauss (one of six, two incorrect) can be turned into a (~~slow~~) numerical method. This is of interest since Cauchy's algorithm, based on "steepest~~est~~ descent", does not always work!